Obstruction Results in Quantization Theory*

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Abstract

Quantization is not a straightforward proposition, as demonstrated by Groenewold's and Van Hove's discovery, exactly fifty years ago, of an "obstruction" to quantization. Their "no-go theorems" assert that it is in principle impossible to consistently quantize every classical observable on the phase space $\mathbf{R}^{2n}$ in a physically meaningful way. A similar obstruction was recently found for S^2, buttressing the common belief that no-go theorems should hold in some generality. Surprisingly, this is not so – it has also just been proven that there is no obstruction to quantizing a torus.

In this paper we take first steps towards delineating the circumstances under which such obstructions will appear, and understanding the mechanisms which produce them. Our objectives are to conjecture a generalized Groenewold-Van Hove theorem, and to determine the maximal subalgebras of observables which can be consistently quantized. This requires a

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study of the structure of Poisson algebras of classical systems and their representations. To these ends we include an exposition of both prequantization (in an extended sense) and quantization theory – formulated in terms of "basic sets of observables," and review in detail the known results for $\mathbf{R}^{2n}$, S^2, and T^2. Our discussion is independent of any particular method of quantization; we concentrate on the structural aspects of quantization theory which are common to all Hilbert space-based quantization techniques.

1 Introduction

Quantization – the problem of constructing the quantum formulation of a system from its classical description – has always been one of the great mysteries of mathematical physics. It is generally acknowledged that quantization is an ill-defined procedure, which cannot be consistently applied to all classical systems. While there is certainly no extant quantization procedure which works well in all circumstances, this assertion nonetheless bears closer scrutiny.

Already from first principles one encounters difficulties. Given that the classical description of a system is an approximation to its quantum description, obtained in a macroscopic limit (when $\hbar \to 0$), one expects that some information is lost in the limit. So quantization should somehow have to compensate for this. But how can a given quantization procedure select, from amongst the myriad of quantum theories all of which have the same classical limit, the physically correct one?

In view of this ambiguity it is not surprising that the many quantization schemes which have been developed over the years – such as the physicists' original "canonical quantization" [**Di**] (and its modern formulations, such as geometric quantization [**Ki**, **So**, **Wo**]), Weyl quantization [**Fo**] (and its successor deformation quantization [**BFFLS**, **Ri2**, **Ri3**]), path integral quantization [**GJ**], and the group theoretic approach to quantization [**Is**], to cite just some – have shortcomings. Rather, is it amazing that they work as well as they do!

But there are deeper, subtler problems, involving the Poisson algebras of classical systems and their representations. In this context the conventional wisdom is that it is impossible to "fully" quantize any given classical system – regardless of the particular method employed – in a way which is consistent with the physicists' Schrödinger quantization of $\mathbf{R}^{2n}$. (We will make this somewhat nebulous statement precise later.) In other words, the assertion is that there exists a universal "obstruction" which forces one to settle for something less than a complete and consistent quantization of *any* system. Each quantization procedure listed above evinces this defect in various examples.

That there are problems in quantizing even simple systems was observed very early on. One difficulty was to identify the analogue of the multiplicative structure of the classical observables in the quantum formalism. For instance, consider the quantization of $\mathbf{R}^{2n}$ with canonical coordinates $\{q^i,\, p_i \,|\, i = 1, \ldots, n\}$,

representing the phase space of a particle moving in $\mathbf{R}^n$. For simple observables the "product $\to$ anti-commutator" rule worked well. But for more complicated observables (say, ones which are quartic polynomials in the positions and momenta), this rule leads to inconsistencies. (See [**AB**, §4], [**Fo**, §1.1] and §4 for a discussion of these factor-ordering ambiguities.) Of course this, in and by itself, might only indicate the necessity of coming up with some subtler symmetrization rule. But attempts to construct a quantization map also conflicted with Dirac's "Poisson bracket $\to$ commutator" rule. This was implicitly acknowledged by Dirac [**Di**, p. 87], where he made the now famous hedge:

> *"The strong analogy between the quantum P.B. ... and the classical P.B. ... leads us to make the assumption that the quantum P.B.s,* or at any rate the simpler ones of them, *have the same values as the corresponding classical P.B.s."*

In any case, as a practical matter, one was forced to limit the quantization to relatively "small" subalgebras of classical observables which could be handled without ambiguity (e.g., polynomials which are at most quadratic in the p's and the q's, or observables which are at most affine functions of the coordinates or of the momenta).

Then, in 1946, Groenewold [**Gr**] showed that the search for an "acceptable" quantization map was futile. The strong version of his "no-go" theorem states that one cannot consistently quantize the Poisson algebra of all polynomials in the q^i and p_i on $\mathbf{R}^{2n}$ as symmetric operators on some Hilbert space $\mathcal{H}$, subject to the requirement that the q^i and p_i be irreducibly represented.[1] Van Hove subsequently refined Groenewold's result [**VH1**]. Thus it is *in principle* impossible to quantize – by *any* means – every classical observable on $\mathbf{R}^{2n}$, or even every polynomial observable, in a way consistent with Schrödinger quantization (which, according to the Stone-Von Neumann theorem, is the import of the irreducibility requirement on the p's and q's). At most, one can consistently quantize certain subalgebras of observables, for instance the ones mentioned in the preceding paragraph.

Of course, Groenewold's remarkable result is valid only for the classical phase space $\mathbf{R}^{2n}$. The immediate problem is to determine whether similar obstructions appear when trying to quantize other symplectic manifolds. Little is known in this regard, and only in the past few years have other examples come to light. Just recently an obstruction was found in the case of the symplectic manifold S^2, representing the (internal) phase space of a massive spinning particle [**GGH**]. It was shown that one cannot consistently quantize the Poisson algebra of spherical harmonics (thought of as "polynomials" in the components S_i of the spin angular momentum vector $\mathbf{S}$), subject to the requirement that the S_i be irreducibly represented on a Hilbert space of dimension greater than one.

[1] There are actually two variants of Groenewold's theorem ("strong" and "weak"); both will be discussed in §4.1.

This is a direct analogue for S^2 of Groenewold's theorem. Combined with the observation that S^2 is in a sense at the opposite extreme from $\mathbf{R}^{2n}$ insofar as symplectic manifolds go, it indicates that no-go theorems can be expected to hold in some generality. But, interestingly enough, they are *not* universal: it is possible to explicitly exhibit a quantization of the full Poisson algebra of the torus T^2 in which a suitable irreducibility requirement is imposed [**Go3**]. An important point, therefore, is to understand the mechanisms which are responsible for these divergent outcomes.

Our goal here is to study such obstruction results for the quantization of a Poisson algebra of a symplectic manifold. We will review the known results in some detail, and give a careful presentation of prequantization (in an extended sense) and quantization, with a view to conjecturing a generalized Groenewold-Van Hove theorem and in particular delineating the circumstances under which it can be expected to hold. Our discussion will be independent of any particular method of quantization; we concentrate on the structural aspects of quantization theory which are common to all Hilbert space-based quantization techniques.

2 Prequantization

Let (M,ω) be a fixed $2n$-dimensional connected symplectic manifold, with associated Poisson algebra $\mathcal{P}(M) := \big(C^\infty(M), \{\cdot\,,\cdot\}\big)$, where $\{\cdot\,,\cdot\}$ is the Poisson bracket. We will abbreviate $\mathcal{P}(M)$ by $\mathcal{P}$ when M is fixed in context.

To start the discussion, we state what it means to "prequantize" a Poisson algebra.

Definition 1 Let $\mathcal{O}$ be any Poisson subalgebra[2] of $\mathcal{P}$ containing the constant function 1. A *prequantization* of $\mathcal{O}$ is a linear map $\mathcal{Q}$ from $\mathcal{O}$ to the linear space $\mathrm{Op}(D)$ of symmetric operators which preserve a fixed dense domain D in some separable Hilbert space $\mathcal{H}$ such that for all $f, g \in \mathcal{O}$

(Q1) $\mathcal{Q}\big(\{f,g\}\big) = \frac{i}{\hbar}\big[\mathcal{Q}(f), \mathcal{Q}(g)\big]$,

(Q2) $\mathcal{Q}(1) = I$, and

(Q3) if the Hamiltonian vector field X_f of f is complete, then $\mathcal{Q}(f)$ is essentially self-adjoint on D.

If $\mathcal{O} = \mathcal{P}$, the prequantization is said to be *full*.

Remarks: 1. By virtue of (Q1) a prequantization $\mathcal{Q}$ of $\mathcal{O}$ is essentially a Lie algebra representation of $\mathcal{O}$ by symmetric operators. (More precisely: if we set $\pi(f) = \frac{i}{\hbar}\,\mathcal{Q}(f)$, then π is a true Lie algebra representation by skew-symmetric operators on D equipped with the commutator bracket. We will blur

[2] By this we mean a linear subspace of $\mathcal{P}$ which is closed under Poisson bracket (but not necessarily under multiplication), i.e., a Lie subalgebra.

the distinction between π and $\mathcal{Q}$.) In this context there are several additional requirements we could place upon $\mathcal{Q}$, such as irreducibility and integrability. However, we do not want to be too selective at this point, so we do not insist on these; they will be discussed as the occasion warrants.

2. Condition (Q2) reflects the fact that if an observable f is a constant c, then the probability of measuring $f = c$ is one regardless of which quantum state the system is in. It also serves to eliminate some "trivial" possibilities, such as the regular representation $f \mapsto X_f$ on $L^2(M, \omega^n)$, where X_f is the Hamiltonian vector field of f.

3. Regarding (Q3), we remark that in contradistinction with Van Hove [**VH1**], we do not confine our considerations to only those classical observables whose Hamiltonian vector fields are complete. Rather than taking the point of view that "incomplete" classical observables cannot be quantized, we simply do not demand that the corresponding quantum operators be essentially self-adjoint ("e.s.a."). We do not imply by this that symmetric operators which are not e.s.a. are acceptable as physical observables; as is well known, this is a controversial point.

4. Notice that no assumptions are made at this stage regarding the multiplicative structure on $\mathcal{O}$ vis-à-vis $\mathcal{Q}$. This is mainly for historical reasons: in classical mechanics the Lie algebra structure has played a more dominant role than the ring structure on $C^\infty(M)$, so it is natural to concentrate on the former. This is also the approach favored by Dirac [**Di**] and the geometric quantization theorists [**So**, **Wo**]. For more algebraic treatments, see [**As**, **Em**, **VN**]. The ring structure is emphasized to a much greater degree in deformation quantization theory [**BFFLS**, **Ri2**, **Ri3**].

Prequantizations in this broad sense (even full ones) are usually easy to construct, cf. [**Ch2**, **Ur**, **Wo**]. Van Hove was the first to construct a full prequantization of $\mathcal{P}(\mathbf{R}^{2n})$ [**VH1**]. It goes as follows: the Hilbert space $\mathcal{H}$ is $L^2(\mathbf{R}^{2n})$, for D we take the Schwartz space $\mathcal{S}(\mathbf{R}^{2n})$ of rapidly decreasing smooth functions (for instance), and for $f \in \mathcal{P}(\mathbf{R}^{2n})$,

$$\mathcal{Q}(f) = -i\hbar \sum_{k=1}^{n} \left[\frac{\partial f}{\partial p_k} \left(\frac{\partial}{\partial q^k} - \frac{i}{\hbar}\, p_k \right) - \frac{\partial f}{\partial q^k} \frac{\partial}{\partial p_k} \right] + f. \tag{1}$$

As luck would have it, however, prequantization representations of the entire Poisson algebra of a symplectic manifold tend to be flawed. For example, the Van Hove prequantization of $\mathcal{P}(\mathbf{R}^{2n})$, when restricted to the Heisenberg subalgebra $\operatorname{span}\{1,\, p_i,\, q^i \,|\, i = 1, \ldots, n\}$, is not unitarily equivalent to the Schrödinger representation (which it ought to be, in the context of a particle moving in $\mathbf{R}^n$ with no superselection rules) [**Bl1**, **Ch1**]. (Recall that the *Schrödinger representation* of the Heisenberg algebra

$$\begin{aligned} \mathrm{h}(2n) \;=\; \operatorname{span}\big\{ P_i,\, Q^i,\, T,\; i = 1, \ldots, n \,\big|\, [P_j, Q^k] = \delta_j^k T,\;\; [P_j, P_k] = 0, \\ [Q^j, Q^k] = 0,\;\; [P_j, T] = 0,\;\; [Q^j, T] = 0 \big\} \end{aligned}$$

is defined to be

$$Q^i \mapsto q^i, \quad P_j \mapsto -i\hbar\,\partial/\partial q^j, \quad \text{and} \quad T \mapsto I \tag{2}$$

on the domain $\mathcal{S}(\mathbf{R}^n) \subset L^2(\mathbf{R}^n)$. It is irreducible in the sense given in the next section.) There are various ways to see this; we give Van Hove's original proof [**VH1**, §17] as it will be useful later. Take $n = 1$ for simplicity. First, define a unitary operator F on $L^2(\mathbf{R}^2)$ by

$$(F\psi)(p,q) = \frac{1}{\sqrt{\hbar}} \int_{-\infty}^{\infty} e^{ipv/\hbar}\psi(v, q-v)\,dv.$$

Then for each fixed $j = 0, 1, \ldots$ take $\mathcal{H}_j$ to be the closure in $L^2(\mathbf{R}^2)$ of the linear span of elements of the form Fh_{jk}, where $h_{jk}(p,q) = h_j(p)h_k(q)$ and h_j is the Hermite function of degree j. Now from (1),

$$\mathcal{Q}(q) = i\hbar\,\frac{\partial}{\partial p} + q, \quad \mathcal{Q}(p) = -i\hbar\,\frac{\partial}{\partial q}.$$

These operators are e.s.a. on $\mathcal{S}(\mathbf{R}^2)$, and one may verify that they strongly commute with the orthogonal projectors onto the closed subspaces $\mathcal{H}_j$.[3] Thus the Van Hove prequantization of $\mathcal{P}(\mathbf{R}^2)$ is reducible when restricted to the Heisenberg subalgebra and hence does not produce the Schrödinger representation. Moreover the association $Fh_{jk}(p,q) \mapsto c_j h_k(q)$, where the c_j are normalization constants, provides a unitary equivalence of each subrepresentation of h(2) on $\mathcal{H}_j$ with the Schrödinger representation on $L^2(\mathbf{R})$, from which we see that the multiplicity of the latter is infinite in the Van Hove representation.

Likewise, the Kostant-Souriau prequantizations of S^2 do not reproduce the familiar spin representations of the unitary algebra u(2). We realize S^2 as a coadjoint orbit of SU(2) in $\mathbf{R}^3$ according to $\mathbf{S}\cdot\mathbf{S} = s^2$, where $\mathbf{S} = (S_1, S_2, S_3)$ is the spin vector and $s > 0$ is the classical spin. It comes equipped with the symplectic form

$$\omega = \frac{1}{2s^2} \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, S_i\, dS_j \wedge dS_k. \tag{3}$$

Now the de Rham class $[\,\omega/h\,]$ is integral iff $s = \frac{n}{2}\hbar$, where n is a positive integer, and the corresponding Kostant-Souriau prequantization line bundles can be shown to be $L^{\otimes n}$ where L is the dual of the universal line bundle over S^2 [**Ki**]. The corresponding prequantum Hilbert spaces $\mathcal{H}_n$ can thus be identified with spaces of square integrable sections ψ of these bundles w.r.t. the inner product

$$\langle \psi, \phi \rangle = \frac{i}{2\pi} \int_{\mathbf{C}} \frac{\overline{\psi(z)}\phi(z)\,dz \wedge d\bar{z}}{(1 + z\bar{z})^{n+2}}$$

[3] Recall that two e.s.a. (or, more generally, normal) operators *strongly commute* iff their spectral resolutions commute, cf. [**ReSi**, §VIII.5]. Two operators A, B *weakly commute* on a domain D if they commute in the ordinary sense, i.e., $[A, B]$ is defined on D and vanishes.

where $z = (S_1 + iS_2)/(s - S_3)$, cf. [**Wo**]. But these $\mathcal{H}_n$ are infinite-dimensional, whereas the standard representation spaces for quantum spin $s = \frac{n}{2}\hbar$ have dimension $n + 1$.

In both examples the prequantization Hilbert spaces are "too big." The main problem is how to remedy this, in other words, how to modify the notion of a prequantization so as to yield a genuine *quantization*.

It is here that the ideas start to diverge, because there is less agreement in the literature as to what constitutes a quantization map. Some versions define it as a prequantization, not necessarily defined on the whole of $\mathcal{P}$, which is irreducible on a "basic set" $\mathcal{B} \subset \mathcal{P}$ [**Ki**]. This is in line with the group theoretical approach to quantization [**Is**], in which context $\mathcal{B}$ is identified with the Lie algebra of a symmetry group;[4] quantization should then yield an irreducible representation of this algebra. For example, when $M = \mathbf{R}^{2n}$, one usually takes $\mathcal{B}$ to be the Heisenberg algebra $\mathrm{h}(2n) = \mathrm{span}\{1,\, p_i,\, q^i \,|\, i = 1, \ldots, n\}$ of polynomials of degree at most one. Similarly, when $M = S^2$, one takes for $\mathcal{B}$ the unitary subalgebra $\mathrm{u}(2) = \mathrm{span}\{1,\, S_1,\, S_2,\, S_3\}$ of spherical harmonics of degree at most one, where S_i are the spin generators. We will plumb in detail the rationale behind these choices of $\mathcal{B}$ in the next section.

A different approach to quantization is to require a prequantization $\mathcal{Q}$ to satisfy some "Von Neumann rule," that is, some given relation between the classical multiplicative structure of $\mathcal{P}$ and operator multiplication on $\mathcal{H}$. (Note that thus far in our discussion the multiplication on $\mathcal{P}$ has been ignored, and it is reasonable to require that quantization preserve at least some of the ring structure of $\mathcal{P}$, given that the Leibniz rule intertwines pointwise multiplication with the Poisson bracket.) There are many different types of such rules [**Co**, **Fo**, **KLZ**, **KS**, **Ku**, **MC**, **VN**], the simplest being of the form:

$$\mathcal{Q}(\varphi \circ f) = \varphi\big(\mathcal{Q}(f)\big) \tag{4}$$

for some distinguished observables $f \in \mathcal{P}$, and certain smooth functions $\varphi \in C^\infty(\mathbf{R})$. (Technically, if φ is not a polynomial, then $\mathcal{Q}(f)$ must be e.s.a. for $\varphi\big(\mathcal{Q}(f)\big)$ to be defined.) In the case of $M = \mathbf{R}^{2n}$, Von Neumann states that the physical interpretation of the quantum theory requires (4) to hold for *all* $f \in \mathcal{P}$ and $\varphi \in C^\infty(\mathbf{R})$ [**VN**]. However, it is easy to see that this is impossible (simple demonstrations are given in [**AB**, **Fo**] as well as §4.1 following); thus the qualifiers in the definition above. In this example, one typically ends up imposing the squaring Von Neumann rule $\varphi(x) = x^2$ on elements x of $\mathrm{h}(2n)$. The relevant rules for the sphere turn out to be somewhat less intuitive; they take the form $\mathcal{Q}\big({S_i}^2\big) = a\mathcal{Q}(S_i)^2 + cI$ for $i = 1, 2, 3$, where a, c are undetermined (representation-dependent) constants subject only to the constraint that $ac \neq 0$. Derivations of these rules in these two examples are given in §4 and [**GGH**].

Another type of quantization is obtained by "polarizing" a prequantization representation [**Wo**]. Following Blattner [**Bl1**], we paraphrase it algebraically as

[4] We typically identify an abstract Lie algebra with its isomorph in $\mathcal{P}$.

follows. Start with a *polarization*, i.e., a maximally commuting Poisson subalgebra $\mathcal{A}$ of $\mathcal{P}$. Then require for the quantization map $\mathcal{Q}$ that the image $\mathcal{Q}(\mathcal{A})$ be "maximally commuting" as operators. If $\mathcal{Q}(\mathcal{A})$ consists of bounded operators, this means that the weak operator closure of the *-algebra generated by $\mathcal{Q}(\mathcal{A})$ $\big(= \mathcal{Q}(\mathcal{A})''\big)$ is maximally commuting in $B(\mathcal{H})$. If $\mathcal{Q}(\mathcal{A})$ contains unbounded operators, one should look for a generating set of normal operators in $\mathcal{Q}(\mathcal{A})$, and require that the Von Neumann algebra generated by their spectral projections is maximally commuting. One can then realize the Hilbert space $\mathcal{H}$ as an L^2-space over the spectrum of this Von Neumann algebra on which this algebra acts as multiplication operators. There will also be a cyclic and separating vector for such an algebra, which provides a suitable candidate for a vacuum vector. Thus another motivation for polarizations is that a maximally commuting set of observables provides a set of compatible measurements, which can determine the state of a system. When $M = \mathbf{R}^{2n}$, one often takes the "vertical" polarization $\mathcal{A} = \big\{f(q^1,\ldots,q^n)\big\}$, in which case one recovers the usual position or coordinate representation. However, in some instances, such as S^2, it is useful to broaden the notion of polarization to that of a maximally commuting subalgebra of the *complexified* Poisson algebra $\mathcal{P}_{\mathbf{C}}$. Then, thinking of S^2 as $\mathbf{C}P^1$, we may take the "antiholomorphic" polarization $\mathcal{A} = \{f(z)\}$, which leads to the usual representations for spin. For treatments of polarizations in the context of deformation quantization, see [**Fr**, **He**].

Thus, informally, a "quantization" could be defined as a prequantization which incorporates one (or more) of the three additional requirements above (or possibly even others). Before proceeding, however, there are two points we would like to make.

The first is that it is of course not enough to simply state the requirements that a quantization map should satisfy; one must also devise methods for implementing them in examples. Thus geometric quantization theory, for instance, provides a specific technique for polarizing certain (Kostant-Souriau) prequantization representations [**Bl1**, **Ki**, **So**, **Wo**]. However, as we are interested here in the structural aspects of quantization theory, and not in specific quantization schemes, we do not attempt to find such implementations.

Second, these three approaches to a quantization map are not independent; there exist subtle connections between them which are not well understood. For instance, demanding that a prequantization be irreducible on some basic set typically leads to the appearance of Von Neumann rules; this is how the Von Neumann rules for $\mathbf{R}^{2n}$ and S^2 mentioned above arise. We will delineate these connections in specific cases in §4, and more generally in §5.

At the core of each of the approaches above is the imposition – in some guise – of an irreducibility requirement, which is used to "cut down" a prequantization representation. Since this is most apparent in the first approach, we will henceforth concentrate on it. We will tie in the two remaining approaches as we go along.

So let $\mathcal{O}$ be a Poisson subalgebra of $\mathcal{P}$, and suppose that $\mathcal{B} \subset \mathcal{O}$ is a "basic set" of observables. Provisionally, we take a *quantization* of the pair $(\mathcal{O}, \mathcal{B})$ to mean a prequantization $\mathcal{Q}$ of $\mathcal{O}$ which (among other things) irreducibly represents $\mathcal{B}$. In the next section we will make this more precise, as well as examine in detail the criteria that $\mathcal{B}$ should satisfy.

Natural issues to address for quantizations are existence, uniqueness and classification, and functoriality. For *pre*quantizations these questions already have partial answers from geometric quantization theory. So for instance we know that if (M, ω) satisfies the integrality condition $[\,\omega/h\,] \in H^2(M, \mathbf{Z})$, then full prequantizations of the Poisson algebra $\mathcal{P}$ exist, and that certain types of these – the Kostant-Souriau prequantizations – can be classified cohomologically [**Ur**, **Wo**]. For some limited types of manifolds the functorial properties of these prequantizations were considered by Blattner [**Bl1**]. However, as there are prequantizations not of the Kostant-Souriau type [**Av**, **Ch2**], these questions are still open in general (especially for manifolds which violate the integrality condition [**We**]).

For quantization maps these questions are far more problematic. Our main focus will be on the existence of *full quantizations*, by which we mean a quantization of $(\mathcal{P}, \mathcal{B})$ for some appropriately chosen basic set $\mathcal{B}$. As indicated earlier, this is poorly understood. In our terminology, the classical (strong) result of Groenewold states that there is no full quantization of $\big(\mathcal{P}(\mathbf{R}^{2n}), \mathrm{h}(2n)\big)$, while the more recent result of [**GGH**] implies essentially the same for $\big(\mathcal{P}(S^2), \mathrm{u}(2)\big)$. On the other hand, nontrivial full quantizations do exist: one can construct such a quantization of T^2 [**Go3**]. However, it does seem that nonexistence results are the rule. In the absence of a full quantization, then, it is important to determine the maximal subalgebras $\mathcal{O}$ of $\mathcal{P}$ for which $(\mathcal{O}, \mathcal{B})$ can be quantized. This we will investigate for $\mathbf{R}^{2n}$ and S^2 in §§4 and 5. At present, questions of uniqueness and classification can only be answered in specific examples.

3 Basic Sets and Quantization

Our first goal here is to make clear what we mean by a basic set of observables $\mathcal{B} \subset \mathcal{P}$. Such sets, in one form or another, play an important role in many quantization methods, such as geometric quantization [**Ki**], deformation quantization [**BFFLS**, **Fr**] and also the group theoretic approach [**Is**].

We start with the most straightforward case, that of an "elementary system" in the terminology of Souriau [**So**, **Wo**]. This means that M is a homogeneous space for a Hamiltonian action of a finite-dimensional Lie group G. The appeal of an elementary system is that it is a classical version of an irreducible representation: using the transitive action of G, one can obtain any classical state from any other one, in direct analogy with the fact that every nonzero vector in a Hilbert space $\mathcal{H}$ is cyclic for an irreducible unitary representation of G on $\mathcal{H}$

[**BaRa**, §5.4]. Now notice that the span $\mathcal{J}$ of the components of the associated (equivariant) momentum map satisfies:

(J1) $\mathcal{J}$ is a finite-dimensional Poisson subalgebra of $\mathcal{P}$,

(J2) the Hamiltonian vector fields $X_f, f \in \mathcal{J}$, are complete, and

(J3) $\{X_f \mid f \in \mathcal{J}\}$ span the tangent spaces to M everywhere.

For both $M = \mathbf{R}^{2n}$ and S^2, the basic sets are precisely of this type: from the elementary systems of the Heisenberg group $\mathrm{H}(2n)$ acting on $\mathbf{R}^{2n}$, and the unitary group $\mathrm{U}(2)$ acting on S^2, we have for $\mathcal{J}$ the spaces $\mathrm{span}\{1, p_i, q^i \mid i = 1, \ldots, n\}$ and $\mathrm{span}\{1, S_1, S_2, S_3\}$, respectively.

Property (J3) is just an infinitesimal restatement of transitivity, and so we call a subset $\mathcal{B}$ of $C^\infty(M)$ *transitive* if it satisfies this condition. Kirillov [**Ki**] uses the terminology "complete," motivated by the fact that such a set of observables locally separates classical states. In this regard, the finite-dimensionality criterion in (J1) plays an important role operationally: it guarantees that a *finite* number of measurements using this collection of observables will suffice to distinguish any two nearby states.

A subset $\mathcal{B} \subset \mathcal{P}$ satisfying (J1)–(J3) is a prototypic basic set. However, there need not exist basic sets in this sense for arbitrary M. For instance, if $M = T^2$, the self-action of the torus is not Hamiltonian – there is no momentum map – and consequently it is difficult to isolate such a basic set. Thinking of T^2 as $\mathbf{R}^2/\mathbf{Z}^2$, a natural choice for $\mathcal{B}$ would be

$$\mathcal{B} = \mathrm{span}\{1, \sin 2\pi x, \cos 2\pi x, \sin 2\pi y, \cos 2\pi y\},$$

but this is not a subalgebra. One alternative would be to consider instead the Poisson algebra $\wp(\mathcal{B})$ generated by $\mathcal{B}$. However, this algebra (viz. the set of trigonometric polynomials) is infinite-dimensional, and there is nothing gained operationally in using an infinite-dimensional algebra to distinguish classical states. Furthermore, as will be shown below, other problems arise if one insists that $\mathcal{B}$ always be a subalgebra of $\mathcal{P}$. We will therefore retain the finite-dimensionality assumption, but merely require that $\mathcal{B}$ be a linear sub*space* as opposed to a Poisson sub*algebra* of $\mathcal{P}$. Thus we make:

Definition 2 A *basic set of observables* $\mathcal{B}$ is a linear subspace of $\mathcal{P}(M)$ such that:

(B1) $\mathcal{B}$ is finite-dimensional,

(B2) the Hamiltonian vector fields $X_f, f \in \mathcal{B}$, are complete,

(B3) $\mathcal{B}$ is transitive,

(B4) $1 \in \mathcal{B}$, and

(B5) $\mathcal{B}$ is a minimal space satisfying these requirements.

We spend some time elaborating on this definition. First, condition (B4) is algebraically natural, as discussed in Remark **2**. (This also explains why, for S^2, we take $\mathcal{B} = \mathrm{u}(2)$ rather than su(2).) Second, the minimality condition (B5) is crucial. From a physical or operational point of view, it is not obvious that this is necessary, as long as $\mathcal{B}$ is finite-dimensional. But the quantization of a pair $(\mathcal{O}, \mathcal{B})$ with $\mathcal{B}$ non-minimal can lead to physically incorrect results.

Here is an example of this phenomenon. First observe that the extended symplectic group HSp($2n$,$\mathbf{R}$) (which is the semi-direct product of the symplectic group Sp($2n$,$\mathbf{R}$) with the Heisenberg group H($2n$)) acts transitively on $\mathbf{R}^{2n}$. This action has a momentum map whose components consist of all inhomogeneous quadratic polynomials in the q^i and p_i. The corresponding subalgebra $\mathcal{J}$ satisfies all the requirements for a basic set save minimality. Now consider again the Van Hove prequantization $\mathcal{Q}$ of $\mathcal{P}(\mathbf{R}^{2n})$ for $n = 1$. In [**VH1**, §17] it is shown that $\mathcal{Q}$ is completely reducible when restricted to the $\mathrm{hsp}(2, \mathbf{R})$-subalgebra $\mathcal{J}$. In fact, there exist exactly two nontrivial HSp(2,$\mathbf{R}$)-invariant closed subspaces $\mathcal{H}_\pm$ in $L^2(\mathbf{R}^2)$, namely (cf. §2)

$$\mathcal{H}_+ = \bigoplus_{j \text{ even}} \mathcal{H}_j \quad \text{and} \quad \mathcal{H}_- = \bigoplus_{j \text{ odd}} \mathcal{H}_j.$$

If we denote the corresponding subrepresentations of $\mathcal{J}$ on $\mathcal{S}(\mathbf{R}^2) \cap \mathcal{H}_\pm$ by $\mathcal{Q}_\pm$, then it follows that $\mathcal{Q}_\pm$ are quantizations of the pair $(\mathcal{J}, \mathcal{J})$. But these quantizations are physically unacceptable, since – just like the full prequantization $\mathcal{Q}$ – they are reducible when further restricted to $\mathrm{h}(2) \subset \mathrm{hsp}(2, \mathbf{R})$. On the one hand, asking for a quantization of $(\mathcal{J}, \mathcal{J})$ in this context is clearly the wrong thing to do, since compatibility with Schrödinger quantization devolves upon the irreducibility of an $\mathrm{h}(2n)$ subalgebra, not an $\mathrm{hsp}(2n, \mathbf{R})$ one. But on the other, this example does illustrate our point.

As well, the minimality requirement reinforces our assertion that it will not do to demand that the basic set be a Poisson subalgebra rather than a linear subspace. For consider again the torus and define the basic sets

$$\mathcal{B}_k = \mathrm{span}\{1, \sin 2\pi kx, \cos 2\pi kx, \sin 2\pi ky, \cos 2\pi ky\}$$

for $k = 1, 2, \ldots$ Each $\mathcal{B}_k$ is a minimal transitive subspace. But $\wp(\mathcal{B}_k)$ is *not* a minimal transitive subalgebra for any k, since $\wp(\mathcal{B}_k) \supset \wp(\mathcal{B}_{2k}) \supset \cdots$. In fact, there probably does not exist a minimal transitive Poisson sub*algebra* of $\mathcal{P}(T^2)$.

Finally, we consider the transitivity requirement (B3). While (B3) is geometrically natural, there are other conditions one might use in place of it. By way of motivation, consider a unitary representation U of a Lie group G on a Hilbert space $\mathcal{H}$. The representation U is irreducible iff the *-algebra $\mathcal{U}$ of bounded operators generated by $\{U(g) \,|\, g \in G\}$ is irreducible, in which case we have the following equivalent characterizations of irreducibility:

(I1) The commutant $\mathcal{U}' = \mathbf{C}I$, and

(I2) the weak operator closure of $\mathcal{U}$ is the algebra of all bounded operators: $\overline{\mathcal{U}}^w = B(\mathcal{H}) \quad (= \mathcal{U}'')$.

That (I1) is equivalent to irreducibility is the content of Schur's Lemma. Property (I2) means that all bounded operators can be built from those in $\mathcal{U}$ by weak operator limits. It follows from (I1), the Von Neumann density theorem [**BrRo**, Cor. 2.4.15], and the fact that $\mathcal{U}' = \left(\overline{\mathcal{U}}^w\right)'$. Clearly (I2) implies (I1).

These restatements of irreducibility have the following classical analogues for a set $\mathcal{F}$ of observables:

(C1) $\{f, g\} = 0$ for all $f \in \mathcal{F}$ implies g is constant, and

(C2) $\mathcal{F}$ generates a dense subspace in $C^\infty(M)$ under linear combinations and pointwise multiplication.

For (C2) a topology on $C^\infty(M)$ must be decided on, and we will use the topology of uniform convergence on compacta of a function as well as its derivatives.

Because the algebraic structures of classical and quantum mechanics are different, (C1) and (C2) lead to inequivalent notions of "classical irreducibility." It is not difficult to verify that (C1) $\Leftarrow$ (B3) $\Leftarrow$ (C2) strictly. In principle either of (C1) or (C2) could serve in place of (B3). Indeed, since on $C^\infty(M)$ one has two algebraic operations, it is natural to consider irreducibility in either context: in terms of the multiplicative structure (C2), or the Poisson bracket (C1). However, it turns out that (C1) is too weak for our purposes, while (C2) is too strong.

The nondegeneracy condition (C1) is equivalent to the statement that observables in $\mathcal{B}$ locally separate states *almost* everywhere [**Ki**]. It is also implied by the statement that the Hamiltonian vector fields of elements of $\mathcal{B}$ span the tangent spaces to M almost everywhere. Consequently, it would not do to replace (B3) by (C1) in the definition of basic set because, e.g., in the case of the sphere span$\{1, S_1, S_2, S_3\}$ would no longer be minimal, which seems both awkward and unreasonable. The same is true for the sets $\mathcal{B}_k$ on the torus, as well as the algebra $\mathrm{sp}(2n, \mathbf{R}) \times \mathbf{R}$ on the homogeneous space $\mathbf{R}^{2n} \setminus \{\mathbf{0}\}$. Condition (C2) is satisfied for the unitary algebra on S^2, the Heisenberg algebra on $\mathbf{R}^{2n}$ and the set $\mathcal{B}_1$ on T^2. But it fails for the symplectic algebra on $\mathbf{R}^{2n} \setminus \{\mathbf{0}\}$ – since the subspace generated by $\mathrm{sp}(2n, \mathbf{R})$ consists of even functions, and also for the sets $\mathcal{B}_k$ with $k > 1$ on the torus – since the subspace they generate consists of doubly periodic functions of period $k > 1$. On the other hand, all these examples satisfy the transitivity requirement (B3), which shows that this is a reasonable condition to impose.

There is still no guarantee in general that sets satisfying (B1)-(B5) will exist for a given symplectic manifold M. However, we can satisfy all conditions except (B2) as follows. Choose an embedding $M \to \mathbf{R}^K$ for some sufficiently large K,

and let $\{f^1, \ldots, f^K\}$ be the restrictions of the standard coordinates on $\mathbf{R}^K$ to M. Then $\mathcal{F} = \mathrm{span}\{1, f^1, \ldots, f^K\}$ satisfies (B1), (B3) and (B4). If $\mathcal{F}$ is not minimal as it stands, one may discard elements of this set until what remains is minimal. If we allow infinite-dimensional basic sets, then localizing $\mathcal{F}$ by means of a partition of unity enables us to satisfy (B2), but now of course (B1) has to be abandoned. In any event, we do know that basic sets will exist whenever (i) M is a Hamiltonian homogeneous space, or (ii) M is compact.

Other properties that basic sets might be required to satisfy are discussed in [**Is**]. For our purposes, (B1)-(B5) will suffice.

We are now ready to discuss what we mean by a "quantization." Let $\mathcal{O}$ be a Poisson subalgebra of $\mathcal{P}$, and suppose that $\mathcal{B} \subset \mathcal{O}$ is a basic set of observables. Two eminently reasonable requirements to place upon a quantization are irreducibility and integrability [**BaRa**, **Fl**, **Is**, **Ki**].

Irreducibility is of course one of the pillars of the quantum theory, and we have already seen the necessity of requiring that quantization represent $\mathcal{B}$ irreducibly. We must however be careful to give a precise definition since the operators $\mathcal{Q}(f)$ are in general unbounded (although, according to (B2) and (Q3), all elements of $\mathcal{Q}(\mathcal{B})$ are e.s.a.). So let $\mathcal{X}$ be a set of e.s.a. operators defined on a common invariant dense domain D in a Hilbert space $\mathcal{H}$. Then $\mathcal{X}$ is *irreducible* provided the only bounded self-adjoint operators which strongly commute with all $X \in \mathcal{X}$ are multiples of the identity. While this definition is fairly standard, and well suited to our needs, we note that other notions of irreducibility can be found in the literature [**BaRa**, **MMSV**].

Given such a set $\mathcal{X}$ of operators, let $\mathcal{U}(\mathcal{X})$ be the *-algebra generated by the unitary operators $\big\{ \exp(it\overline{X}) \,|\, t \in \mathbf{R},\ X \in \mathcal{X} \big\}$, where $\overline{X}$ is the closure of X. Then by Schur's Lemma $\mathcal{X}$ is irreducible iff the only closed subspaces of $\mathcal{H}$ which are invariant under $\mathcal{U}(\mathcal{X})$ are $\{0\}$ and $\mathcal{H}$.

Turning now to integrability, we first consider the case when the Poisson algebra $\wp(\mathcal{B})$ generated by the basic set is finite-dimensional. Then it is natural to demand that the Lie algebra representation $\mathcal{Q}\big(\wp(\mathcal{B})\big)$ on $D \subset \mathcal{H}$ be *integrable* in the following sense: there exists a unitary representation Π of some Lie group with Lie algebra (isomorphic to) $\wp(\mathcal{B})$ on $\mathcal{H}$ such that $d\Pi(f)|D = \mathcal{Q}(f)$ for all $f \in \wp(\mathcal{B})$, where $d\Pi$ is the derived representation of Π. For this it is neither necessary *nor* sufficient that elements of $\wp(\mathcal{B})$ quantize to e.s.a. operators on D.[5] But integrability will follow from the following result of Flato et al., cf. [**Fl**] and [**BaRa**, Ch. 11].

Proposition 1 *Let $\mathcal{G}$ be a real finite-dimensional Lie algebra, and let π be a representation of $\mathcal{G}$ by skew-symmetric operators on a common dense invariant domain D in a Hilbert space $\mathcal{H}$. Suppose that $\{\xi_1, \ldots, \xi_k\}$ generates $\mathcal{G}$ by linear combinations and repeated brackets. If D contains a dense set of separately analytic vectors for $\big\{\pi(\xi_1), \ldots, \pi(\xi_k)\big\}$, then there exists a unique unitary rep-*

[5] See Remark **8** following.

resentation Π of the connected simply connected Lie group with Lie algebra $\mathcal{G}$ on $\mathcal{H}$ such that $d\Pi(\xi)|D = \pi(\xi)$ for all $\xi \in \mathcal{G}$.

(We recall that if X is an operator on $\mathcal{H}$, a vector ψ is *analytic* for X provided the series

$$\sum_{k=0}^{\infty} \frac{\|X^k\psi\|}{k!}\, t^k$$

is defined and converges for some $t > 0$. If $\{X_1, \ldots, X_k\}$ is a set of operators defined on a common invariant dense domain D, a vector $\psi \in D$ is *separately analytic* for $\{X_1, \ldots, X_k\}$ if ψ is analytic for each X_j. By a slight abuse of terminology, we will say that a vector is separately analytic for a linear space of operators $\mathcal{X}$ if it is separately analytic for some basis $\{X_1, \ldots, X_k\}$ of $\mathcal{X}$.)

However, it may happen that $\wp(\mathcal{B})$ is not finite-dimensional (as in the case of the torus). Then there need not exist an (infinite-dimensional) Lie group having $\wp(\mathcal{B})$ as its Lie algebra. Even if such a Lie group existed, integrability is far from automatic, and technical difficulties abound. Thus we will not insist that a quantization be integrable in general. On the other hand, the analyticity requirement in Proposition 1 makes sense under all circumstances,[6] and does guarantee integrability when $\wp(\mathcal{B})$ is "nice," so we will adopt it in lieu of integrability.

Therefore we have at last:

Definition 3 A *quantization* of the pair $(\mathcal{O}, \mathcal{B})$ is a prequantization $\mathcal{Q}$ of $\mathcal{O}$ on $\mathrm{Op}(D)$ satisfying

(Q4) $\mathcal{Q}(\mathcal{B}) = \{\mathcal{Q}(f) \mid f \in \mathcal{B}\}$ is an irreducible set, and

(Q5) D contains a dense set of separately analytic vectors for $\mathcal{Q}(\mathcal{B})$.

A quantization $\mathcal{Q}$ is *nontrivial* if the representation space is neither one- nor zero-dimensional.

Remarks: 5. There are a number of analyticity assumptions similar to (Q5) that one could make [**Fl**]; we have chosen the weakest possible one.

6. (Q5) is not a severe restriction: when $\wp(\mathcal{B})$ is finite-dimensional, it is always possible to find representations of it on domains D which satisfy this property [**Fl**]. On the other hand, nonintegrable representations do exist in general [**Fl**, p. 247].

7. Proposition 1 requires that a specific generating set for $\mathcal{Q}\big(\wp(\mathcal{B})\big)$ be singled out. This also is not a severe restriction: in examples, $\mathcal{B}$ is usually given as the linear span of such a set. It is possible that (Q5) could be satisfied for one such set but not another, but Remark **6** shows that the domain D can be chosen in such a way that this cannot happen if $\wp(\mathcal{B})$ is finite-dimensional.

[6] As long as $\wp(\mathcal{B})$ is finitely generated, which is assured by (B1).

8. With regard to essential self-adjointness vis-à-vis integrability, we observe that the operators $\mathcal{Q}(f)$ for $f \in \wp(\mathcal{B})$ need *not* be e.s.a. on D; Proposition 1 only guarantees that they have s.a. extensions. This is consistent with the fact that the Hamiltonian vector fields of elements of $\wp(\mathcal{B})$ are not necessarily complete. When f is complete (e.g., $f \in \mathcal{B}$), (Q3) requires that $\mathcal{Q}(f)$ be e.s.a. and the proposition correspondingly yields $d\Pi(f) = \overline{\mathcal{Q}(f)}$. That essential self-adjointness alone is not sufficient to guarantee integrability is well known, cf. [**ReSi**, §VIII.5].

9. It is important to note that irreducibility does not imply integrability. For instance, there is an irreducible representation of h(2) which is not integrable [**ReSi**, p. 275].

We end this section with a brief comment on the domains D appearing in Definition 3. For a representation π of a Lie algebra $\mathcal{G}$ on a Hilbert space $\mathcal{H}$, there is typically a multitude of common, invariant dense domains that one can use as carriers of the representation. (See [**BaRa**, §11.2] for a discussion of some of the possibilities.) But what is ultimately important for our purposes are the closures $\overline{\pi(\xi)}$ for $\xi \in \mathcal{G}$, and not the $\pi(\xi)$ themselves. So we do not want to distinguish between two representations π on $\mathrm{Op}(D)$ and π' on $\mathrm{Op}(D')$ whenever $\overline{\pi(\xi)} = \overline{\pi'(\xi)}$, in which case we say that π and π' are *coextensive*. In particular, it may happen that the given domain D for a representation π does not satisfy (Q5), but there is an extension to a coextensive representation π' on a domain D' that does.[7] In such cases we will suppose that the representation has been so extended.

4 Examples

In this section we present the gist of the arguments that there are no nontrivial quantizations of either $\big(\mathcal{P}(\mathbf{R}^{2n}), \mathrm{h}(2n)\big)$ or $\big(\mathcal{P}(S^2), \mathrm{u}(2)\big)$. The complete proofs can be found in [**AM**, **Ch1**, **Fo**, **Go1**, **Gr**, **GS**, **VH1**, **VH2**] for $\mathbf{R}^{2n}$ and in [**GGH**] for S^2. In both cases the detailed structure of the Poisson algebras and their representation theory is used, which makes it hard to generalize these results to other symplectic manifolds. Finally, we show following [**Go3**] that there is a full quantization of $\big(\mathcal{P}(T^2), \mathcal{B}_k\big)$ for each integer $k > 0$, where $\mathcal{B}_k$ is the basic set defined in the last section. We also take this opportunity to point out a defect in the standard presentations of the Groenewold-Van Hove theorem for $\mathbf{R}^{2n}$.

4.1 $\mathbf{R}^{2n}$

Before proceeding with the no-go theorem for $\mathbf{R}^{2n}$, we remark that already at a purely mathematical level one can observe a suggestive structural mismatch

[7] A simple illustration is provided by the Schrödinger representation (2) with $D = C_0^\infty(\mathbf{R}^n)$ and $D' = \mathcal{S}(\mathbf{R}^n)$.

between the classical and the quantum formalisms. Since a prequantization is essentially a Lie algebra representation, it "compares" the Poisson algebra structure of $\mathcal{P}(\mathbf{R}^{2n})$ with the Lie algebra of (skew-) symmetric operators (preserving a dense domain D) equipped with the commutator bracket. But if we take $P \subset \mathcal{P}(\mathbf{R}^{2n})$ to be the subalgebra of polynomials, Joseph [**Jo**] has shown that P has outer derivations, but the enveloping algebra of the Heisenberg algebra $\mathrm{h}(2n)$ – and hence that of the Schrödinger representation thereof on $L^2(\mathbf{R}^n)$ – has none.

Furthermore, one can see at the outset that it is impossible for a prequantization to satisfy the "product → anti-commutator" rule. Taking $n = 1$ for simplicity, suppose $\mathcal{Q}$ were a prequantization of the polynomial subalgebra $P \subset \mathcal{P}(\mathbf{R}^2)$ for which

$$\mathcal{Q}(fg) = \tfrac{1}{2}\big(\mathcal{Q}(f)\mathcal{Q}(g) + \mathcal{Q}(g)\mathcal{Q}(f)\big) \tag{5}$$

for all $f, g \in P$. Take $f(p,q) = p$ and $g(p,q) = q$. Then

$$\begin{aligned}
\tfrac{1}{4}\big(\mathcal{Q}(p)\mathcal{Q}(q) + \mathcal{Q}(q)\mathcal{Q}(p)\big)^2 \;&=\; \mathcal{Q}(pq)^2 \\
&=\; \mathcal{Q}\big(p^2q^2\big) \;=\; \tfrac{1}{2}\big(\mathcal{Q}(p)^2\mathcal{Q}(q)^2 + \mathcal{Q}(q)^2\mathcal{Q}(p)^2\big).
\end{aligned}$$

Now by (Q1) we have $[\mathcal{Q}(p), \mathcal{Q}(q)] = -i\hbar$, so that the L.H.S. reduces to

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\hbar\mathcal{Q}(q)\mathcal{Q}(p) - \tfrac{1}{4}\hbar^2 I$$

while the R.H.S. becomes

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\hbar\mathcal{Q}(q)\mathcal{Q}(p) - \hbar^2 I.$$

As the product → anti-commutator rule is equivalent to the squaring Von Neumann rule $\mathcal{Q}\big(f^2\big) = \mathcal{Q}(f)^2$, this contradiction also shows that the latter is inconsistent with prequantization. Note that the contradiction is obtained on quartic polynomials; there is no problem if consideration is limited to observables which are at most cubic.

This argument only used axiom (Q1) in the specific instance $[\mathcal{Q}(p), \mathcal{Q}(q)] = -i\hbar I$. Consequently, one still obtains a contradiction if one drops (Q1) and instead insists that $\mathcal{Q}$ be consistent with Schrödinger quantization (in which context this one commutation relation remains valid, cf. (2)). This *manifest* impossibility of satisfying the product → anti-commutator rule while being consistent with Schrödinger quantization is one reason we have decided to concentrate on the Lie structure as opposed to the multiplicative structure of $C^\infty(M)$. See [**AB**] for further results in this direction.

We now turn to the no-go theorem for $\mathbf{R}^{2n}$. We shall state the main results for $\mathbf{R}^{2n}$ but, for convenience, usually prove them only for $n = 1$. The proofs for higher dimensions are immediate generalizations of these. In what follows P denotes the subalgebra of polynomials, P^k the subspace of polynomials of degree at most k and P_k the space of homogeneous polynomials of degree k.

Note that $P^1 \cong \mathrm{h}(2n)$, $P_2 \cong \mathrm{sp}(2n, \mathbf{R})$, and $P^2 \cong \mathrm{hsp}(2n, \mathbf{R})$, where the latter is the Lie algebra of the extended symplectic group, cf. §3.

There are actually several versions of the Groenewold-Van Hove no-go theorem, depending upon the properties one wants a quantization to satisfy. We begin with the weakest result, which requires no assumptions on $\mathcal{Q}$ beyond those given in Definition 3.

We first observe that there *does* exist a quantization $d\varpi$ of the pair (P^2, P^1). For $n = 1$ it is given by the familiar formulæ

$$d\varpi(q) = q, \qquad d\varpi(1) = I, \qquad d\varpi(p) = -i\hbar\frac{\partial}{\partial q} \tag{6}$$

$$d\varpi(q^2) = q^2, \quad d\varpi(pq) = -i\hbar\left(q\frac{\partial}{\partial q} + \frac{1}{2}\right), \quad d\varpi(p^2) = -\hbar^2\frac{\partial^2}{\partial q^2} \tag{7}$$

on the domain $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R})$. Properties (Q1)–(Q3) are readily verified. (Q4) follows automatically since the restriction of $d\varpi$ to P^1 is just the Schrödinger representation. For (Q5), we recall the well-known fact that the Hermite functions form a dense set of separately analytic vectors for $d\varpi(P^1)$. Since these functions are also separately analytic vectors for $d\varpi(P_2)$ [**Fo**, Prop. 4.49], the operator algebra $d\varpi(P^2)$ is integrable to a representation ϖ of the universal cover $\widetilde{\mathrm{HSp}}(2n, \mathbf{R})$ of $\mathrm{HSp}(2n, \mathbf{R})$[8] (thereby justifying our notation "$d\varpi$"). ϖ is known as the "extended metaplectic representation"; detailed discussions of it may be found in [**Fo**, **GS**].

We call $d\varpi$ the "extended metaplectic quantization." It has the following crucial property.

Proposition 2 *The extended metaplectic quantization is the* unique *quantization of* $\big(\mathrm{hsp}(2n, \mathbf{R}), \mathrm{h}(2n)\big)$ *which exponentiates to a unitary representation of* $\widetilde{\mathrm{HSp}}(2n, \mathbf{R})$.

By "unique," we mean up to unitary equivalence and coextension of representations (as explained at the end of §3).

Proof: Suppose $\mathcal{Q}$ is a another such quantization of $\big(\mathrm{hsp}(2n, \mathbf{R}), \mathrm{h}(2n)\big)$ on some Hilbert space $\mathcal{H}$. Then $\mathcal{Q}\big(\mathrm{hsp}(2n, \mathbf{R})\big)$ can be integrated to a representation τ of $\widetilde{\mathrm{HSp}}(2n, \mathbf{R})$, and (Q4) implies that τ, when restricted to $\mathrm{H}(2n) \subset \widetilde{\mathrm{HSp}}(2n, \mathbf{R})$, is irreducible. The Stone-Von Neumann Theorem then states that this representation of $\mathrm{H}(2n)$ is unitarily equivalent to the Schrödinger representation, and hence $\tau = U\varpi U^{-1}$ for some unitary map $U : L^2(\mathbf{R}^n) \to \mathcal{H}$ by [**Fo**, Prop. 4.58]. Consequently, $\mathcal{Q}(f) = U\overline{d\varpi(f)}U^{-1}|D$ for all $f \in \mathrm{hsp}(2n, \mathbf{R})$. Since the Hamiltonian vector fields of such f are complete, the corresponding operators $\mathcal{Q}(f)$ are e.s.a., and therefore $\overline{\mathcal{Q}(f)}$ and $U\overline{d\varpi(f)}U^{-1}$ are coextensive. □

The first, and weakest version of the no-go theorem is:

[8] This representation actually drops to the double cover of $\mathrm{HSp}(2n, \mathbf{R})$, but we do not need this fact here.

Theorem 3 (Weak No-Go Theorem) *The extended metaplectic quantization of* (P^2, P^1) *cannot be extended beyond* P^2 *in* P.

Since P^2 is a maximal Poisson subalgebra of P [**GS**, §16], we may restate this as: *There exists no quantization of* (P, P^1) *which reduces to the extended metaplectic quantization on* P^2.

Proof: Let $\mathcal{Q}$ be a quantization of (P, P^1) which extends the metaplectic quantization of (P^2, P^1). We will show that a contradiction arises when cubic polynomials are considered.

By inspection of (6) and (7), we see that the product $\rightarrow$ anti-commutator rule (5) is valid for $f,\, g \in P^1$. In particular, we have the Von Neumann rules

$$\mathcal{Q}(q^2) = \mathcal{Q}(q)^2, \;\; \mathcal{Q}(p^2) = \mathcal{Q}(p)^2 \tag{8}$$

and

$$\mathcal{Q}(qp) = \tfrac{1}{2}\big(\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)\big). \tag{9}$$

These in turn lead to "higher degree Von Neumann rules."

Lemma 1 *For all real-valued polynomials* r,

$$\mathcal{Q}\big(r(q)\big) = r\big(\mathcal{Q}(q)\big), \quad \mathcal{Q}\big(r(p)\big) = r\big(\mathcal{Q}(p)\big),$$

$$\mathcal{Q}\big(r(q)p\big) = \tfrac{1}{2}\big[r\big(\mathcal{Q}(q)\big)\mathcal{Q}(p) + \mathcal{Q}(p)r\big(\mathcal{Q}(q)\big)\big],$$

and

$$\mathcal{Q}\big(qr(p)\big) = \tfrac{1}{2}\big[\mathcal{Q}(q)r\big(\mathcal{Q}(p)\big) + r\big(\mathcal{Q}(p)\big)\mathcal{Q}(q)\big].$$

Proof: We illustrate this for $r(q) = q^3$. The other rules follow similarly using induction. Now $\{q^3, q\} = 0$ whence by (Q1) we have $\big[\mathcal{Q}(q^3), \mathcal{Q}(q)\big] = 0$. Since also $\big[\mathcal{Q}(q)^3, \mathcal{Q}(q)\big] = 0$, we may write $\mathcal{Q}(q^3) = \mathcal{Q}(q)^3 + T$ for some operator T which (weakly) commutes with $\mathcal{Q}(q)$. We likewise have

$$\big[\mathcal{Q}(q^3), \mathcal{Q}(p)\big] = -i\hbar\, \mathcal{Q}\big(\{q^3, p\}\big) = 3i\hbar\, \mathcal{Q}(q^2) = 3i\hbar\, \mathcal{Q}(q)^2 = \big[\mathcal{Q}(q)^3, \mathcal{Q}(p)\big]$$

from which we see that T commutes with $\mathcal{Q}(p)$ as well. Consequently, T also commutes with $\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)$. But then from (9),

$$\begin{aligned}
\mathcal{Q}(q^3) &= \tfrac{1}{3}\, \mathcal{Q}\big(\{pq, q^3\}\big) = \tfrac{i}{3\hbar}\, \big[\mathcal{Q}(pq), \mathcal{Q}(q^3)\big] \\
&= \tfrac{i}{3\hbar}\, \big[\tfrac{1}{2}\big(\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)\big), \mathcal{Q}(q)^3 + T\big] \\
&= \tfrac{i}{6\hbar}\, \big[\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q), \mathcal{Q}(q)^3\big] = \mathcal{Q}(q)^3. \quad \triangledown
\end{aligned}$$

With this lemma in hand, it is now a simple matter to prove the no-go theorem. Consider the classical equality

$$\tfrac{1}{9}\{q^3, p^3\} = \tfrac{1}{3}\{q^2p, p^2q\}.$$

Quantizing and then simplifying this, the formulæ in Lemma 1 give

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\mathcal{Q}(q)\mathcal{Q}(p) - \tfrac{2}{3}I$$

for the L.H.S., and

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\mathcal{Q}(q)\mathcal{Q}(p) - \tfrac{1}{3}I$$

for the R.H.S., which is a contradiction. □

In Groenewold's paper [**Gr**] a much stronger result was claimed; in our terminology, his assertion was that there is no quantization of (P, P^1). This is not what Theorem 3 states. For if $\mathcal{Q}$ is a quantization of (P, P^1), then while of course $\mathcal{Q}(P^1)$ must coincide with Schrödinger quantization, $\mathcal{Q}$ need *not* be the extended metaplectic quantization when restricted to P^2. The problem is that $\mathcal{Q}(P^2)$ is not necessarily integrable; (Q5) only guarantees that $\mathcal{Q}(P^1)$ can be integrated.

With an extra assumption which will guarantee the integrability of $\mathcal{Q}(P^2)$, it is therefore possible to obtain a "true" no-go result. Van Hove supplied such an assumption, which in particular implies: if the Hamiltonian vector fields of f, g are complete and $\{f, g\} = 0$, then $\mathcal{Q}(f)$ and $\mathcal{Q}(g)$ *strongly* commute. In our view, however, his assumption is ad hoc, and stronger than what one actually needs (or wants). We find it preferable to enforce the integrability of $\mathcal{Q}(P^2)$ in a more direct manner. Noting that P^2 is the Poisson normalizer of P^1 in P, we strengthen (Q5) as follows. Given a basic set $\mathcal{B} \subseteq \mathcal{O}$, let $\mathcal{N}_{\mathcal{O}}\big(\wp(\mathcal{B})\big) := \mathcal{N}\big(\wp(\mathcal{B})\big) \cap \mathcal{O}$ be the normalizer of $\wp(\mathcal{B})$ *in* $\mathcal{O}$ $\big($where $\mathcal{N}\big(\wp(\mathcal{B})\big)$ denotes the normalizer of $\wp(\mathcal{B})$ in $\mathcal{P}\big)$, and set

(Q5′) D contains a dense set of separately analytic vectors for (some Lie generating basis of) $\mathcal{Q}\big(\mathcal{N}_{\mathcal{O}}(\wp(\mathcal{B}))\big)$.

We call a quantization satisfying (Q5′) in place of (Q5) *strong*. In view of Remark **6** (Q5′) is not a severe restriction mathematically, provided $\mathcal{N}_{\mathcal{O}}\big(\wp(\mathcal{B})\big)$ is finite-dimensional, and the alternative – which ultimately results in a nonintegrable quantization of $\mathcal{N}_{\mathcal{O}}\big(\wp(\mathcal{B})\big)$ – is clearly pathological. Moreover, as we will see, $\mathcal{N}_{\mathcal{O}}\big(\wp(\mathcal{B})\big)$ plays an important role in determining the maximal Poisson subalgebras which are quantizable, and so it is natural to single it out in this manner. In any case, with (Q5′) we are now able to state a strong no-go result.

Theorem 4 (Strong No-Go Theorem) *There does not exist a strong quantization of* $\big(\mathcal{P}(\mathbf{R}^{2n}), \mathrm{h}(2n)\big)$.

Proof: Actually, we will prove a sharper result, viz., there exists no strong quantization of (P, P^1). If $\mathcal{Q}$ were such a quantization, then as P^2 is finite-dimensional (Q5$'$) implies that $\mathcal{Q}(P^2)$ is integrable and, as before, (Q4) implies that $\mathcal{Q}(P^1)$ is equivalent to the Schrödinger representation. Then Proposition 2 shows that $\mathcal{Q}(P^2)$ must be equivalent to the extended metaplectic quantization, and this contradicts Theorem 3. The Theorem now follows from this result and the fact that the normalizer of P^1 in $\mathcal{P}$ is the same as its normalizer in P, viz. P^2. $\square$

Van Hove [**VH1**] actually gave a more refined argument for Theorem 4 using in his analysis only those observables $f \in \mathcal{P}$ with complete Hamiltonian vector fields, and still obtained a contradiction from $\mathcal{Q}$.

Finally, we hasten to add that there are subalgebras of P other than P^2 which can be quantized. For example, let

$$S = \left\{ \sum_{i=1}^{n} f^i(q) p_i + g(q) \right\},$$

where f^i and g are polynomials. Then it is straightforward to verify that $\sigma : S \to \mathrm{Op}(\mathcal{S}(\mathbf{R}^n))$ given by

$$\sigma\big(f(q)p + g(q)\big) = -i\hbar \left(f(q)\frac{d}{dq} + \frac{1}{2} f'(q) \right) + g(q) \tag{10}$$

(for $n = 1$) is a (strong) quantization of (S, P^1). σ is the familiar "position" or "coordinate representation" in quantum mechanics. Since S is also a maximal subalgebra of P, we are able to prove the following analogue of Theorem 3.

Theorem 5 *The position representation σ of (S, P^1) cannot be extended beyond S in P.*

Proof: Suppose $\mathcal{Q}$ were a quantization of (P, P^1) which extends σ. (Since S is maximal in P, (Q1) implies that any quantization which extends σ must be defined on all of P.) Consider $\mathcal{Q}(p^2)$. Mimicking the arguments in the proof of Lemma 1, we find that $\mathcal{Q}(p^2) = \mathcal{Q}(p)^2 + T$, where T commutes with both $\mathcal{Q}(q) = q$ and $\mathcal{Q}(p) = -i\hbar\frac{d}{dq}$. Quantizing the classical relation $2p^2 = \{p^2, pq\}$, we may then show using (10) that $T = 0$, and so $\mathcal{Q}$ restricts to the extended metaplectic representation on P^2. Thus $\mathcal{Q}$ must also extend $d\varpi$ and the result now follows from Theorem 3. $\square$

It unfortunately does not seem possible to prove a uniqueness result for σ analogous to Proposition 2. The reason stems in part from the fact that $\mathcal{N}_S(P^1) = \mathrm{span}\{1, p, q, q^2, pq\}$ does not satisfy $\big\{\mathcal{N}_S(P^1), \mathcal{N}_S(P^1)\big\} = \mathcal{N}_S(P^1)$, cf. [**Fo**, Ch. 4].

A similar analysis applies to the the Fourier transform of the subalgebra S, i.e., the "momentum" subalgebra of all polynomials which are at most affine in

the coordinates q^i. In fact, it is not difficult to see that P^2, S and its Fourier transform exhaust the list of maximal subalgebras of P which contain P^1.

4.2 S^2

Now we turn our attention to the sphere. Since S^2 is compact, all classical observables are complete. Moreover, the basic set $\mathrm{u}(2) = \mathrm{span}\{1, S_1, S_2, S_3\}$ is a compact Lie algebra (cf. §5). Consequently all the functional analytic difficulties present in the case of $\mathbf{R}^{2n}$ disappear. But the actual computations, which were fairly routine for $\mathbf{R}^{2n}$, turn out to be much more complicated for S^2.

The Poisson bracket on $C^\infty(S^2)$ corresponding to (3) is

$$\{f, g\} = -\sum_{i,j,k=1}^{3} \epsilon_{ijk}\, S_i\, \frac{\partial f}{\partial S_j} \frac{\partial g}{\partial S_k}.$$

In particular, we have the relations $\{S_j, S_k\} = -\sum_{l=1}^{3} \epsilon_{jkl}\, S_l$. Let H_k denote the space of spherical harmonics of degree k, and define $P^k = \oplus_{l=0}^{k} H_l$, where the orthogonal direct sum is given by harmonic decomposition. We may identify elements f of P^k with polynomials of degree at most k in the components S_i of the spin vector, subject to the relation

$${S_1}^2 + {S_2}^2 + {S_3}^2 = 1.$$

(By the "degree" of such an f we mean the minimum degree of its representatives.) Note that $P^1 \cong \mathrm{u}(2)$. Set $P = \oplus_{k=0}^{\infty} H_k$.

Let $\mathcal{Q}$ be a quantization of $\big(\mathcal{P}(S^2), \mathrm{u}(2)\big)$ on a Hilbert space $\mathcal{H}$, whence

$$\big[\mathcal{Q}(S_j), \mathcal{Q}(S_k)\big] = i\hbar \sum_{l=0}^{3} \epsilon_{jkl}\, \mathcal{Q}(S_l) \tag{11}$$

and

$$\mathcal{Q}(\mathbf{S}^2) = s^2 I. \tag{12}$$

By (Q5) and Proposition 1, $\mathcal{Q}\big(\mathrm{u}(2)\big)$ can be exponentiated to a unitary representation of the universal cover $\mathrm{SU}(2) \times \mathbf{R}$ of $\mathrm{U}(2)$ which, according to (Q4), is irreducible. Therefore $\mathcal{H}$ must be finite-dimensional, and $\mathcal{Q}\big(\mathrm{u}(2)\big)$ must be one of the usual spin angular momentum representations, labeled by $j = 0, \frac{1}{2}, 1, \ldots$ For a fixed value of j, $\dim \mathcal{H} = 2j + 1$ and

$$\sum_{i=1}^{3} \mathcal{Q}(S_i)^2 = \hbar^2 j(j+1) I. \tag{13}$$

Our goal is show that no such (nontrivial) quantization exists. Patterning our analysis after that for $\mathbf{R}^{2n}$, we use irreducibility to derive some generalized Von Neumann rules.

Lemma 2 *For $i = 1, 2, 3$ we have*

$$\mathcal{Q}\big({S_i}^2\big) = a\mathcal{Q}(S_i)^2 + cI \tag{14}$$

where a, c are representation dependent real constants with $ac \neq 0$.

The proof is in [**GGH**]. From this we also derive

$$\mathcal{Q}(S_i S_k) = \frac{a}{2}\big(\mathcal{Q}(S_i)\mathcal{Q}(S_k) + \mathcal{Q}(S_k)\mathcal{Q}(S_i)\big) \tag{15}$$

for $i \neq k$. (As an aside, these formulæ show that a quantization, if it exists, is badly behaved with respect to the multiplicative structure on $C^\infty(S^2)$; in particular, the product $\to$ anti-commutator rule cannot hold. Remarkably, this is as it should be: for *if* this rule were valid, then – subject to a few mild assumptions on $\mathcal{Q}$ – the classical spectrum of S_3, say, would have to coincide with that of $\mathcal{Q}(S_3)$ which is contrary to experiment [**GGH**].) With these tools, we can now prove the main result:

Theorem 6 *There is no nontrivial quantization of $\big(\mathcal{P}(S^2), \mathrm{u}(2)\big)$.*

Proof: As for Theorem 4, we will actually prove a sharper result, viz. there is no nontrivial quantization of (P, P^1).

Fix $j > 0$, as $j = 0$ produces a trivial quantization. Assuming that $\mathcal{Q}$ is a quantization of (P, P^1), we can use (11)-(15) to quantize the classical relation

$$s^2 S_3 = \big\{{S_1}^2 - {S_2}^2,\, S_1 S_2\big\} - \big\{S_2 S_3,\, S_3 S_1\big\},$$

thereby obtaining

$$s^2 = a^2\hbar^2\big(j(j+1) - \tfrac{3}{4}\big) \tag{16}$$

which contradicts $s > 0$ for $j = \frac{1}{2}$. Now assume $j > \frac{1}{2}$, and quantize

$$2s^2 S_2 S_3 = \big\{{S_2}^2, \{S_1 S_2, S_1 S_3\}\big\} - \tfrac{3}{4}\big\{{S_1}^2, \{{S_1}^2, S_2 S_3\}\big\},$$

similarly obtaining

$$s^2 = a^2\hbar^2\big(j(j+1) - \tfrac{9}{4}\big)$$

which contradicts (16). Thus we have derived contradictions for all $j > 0$, and the theorem is proven. □

In view of the impossibility of quantizing (P, P^1), one can ask what the maximal subalgebras in P are to which we can extend an irreducible representation of P^1. The following chain of results, which we quote without proof (cf. [**GGH**]), provides the answer.

Proposition 7 *P^1 is a maximal Poisson subalgebra of $O \oplus \mathbf{R} \subset P$, where O is the Poisson algebra consisting of polynomials containing only terms of odd degree.*

Next we establish a no-go theorem for $(O \oplus \mathbf{R}, P^1)$. However, the generalized Von Neumann rules listed in Lemma 2 involve only even degree polynomials, so these are not applicable in O. Fortunately, we have another set of generalized Von Neumann rules, also implied by the irreducibility of $\mathcal{Q}(P^1)$, involving only terms of odd degree.

Lemma 3 *If $\mathcal{Q}$ is a quantization of $(O \oplus \mathbf{R}, P^1)$, then for $i = 1, 2, 3$,*

$$\mathcal{Q}\big({S_i}^3\big) = b\mathcal{Q}(S_i)^3 + e\mathcal{Q}(S_i)$$

where $b, e \in \mathbf{R}$.

From this we prove (with far greater effort):

Theorem 8 *There is no nontrivial quantization of $(O \oplus \mathbf{R}, P^1)$.*

Now $O \oplus \mathbf{R}$ is itself a maximal subalgebra of P, and in fact the only Poisson subalgebras of P strictly containing P^1 are $O \oplus \mathbf{R}$ and P itself. Thus Theorem 8 and Proposition 7 combine to yield our sharpest result for the sphere:

Corollary 1 *No nontrivial quantization of P^1 can be extended to a larger subalgebra of P.*

An interesting observation is that there is *no* obstruction for the quantum spin $j = 0$ representation. In fact, there exists a (unique) trivial quantization of $(\mathcal{P}, P^1)$ with $\mathcal{Q}(S_i) = 0$, but $\mathcal{Q}\big({S_i}^2\big) = \frac{s^2}{3} I$ for all i. It is defined by $Q(f) = f_0$, where f_0 is the constant term in the harmonic decomposition of $f \in \mathcal{P}$.

There are crucial structural differences between the Groenewold-Van Hove analysis of $\mathbf{R}^2$ and the current analysis of S^2. Within P, span$\{1, p, q\}$ has as its Poisson normalizer the algebra of polynomials at most degree 2, and there is no obstruction to quantization in this algebra: the obstruction comes from the cubic polynomials. On the other hand, for the sphere, the algebra span$\{1, S_1, S_2, S_3\}$ is self-normalizing in P; we obtain an obstruction in the quadratic polynomials, and find that there is no extension possible for a quantization of P^1. The fact that this u(2)-subalgebra is self-normalizing is one reason why we are able to obtain "strong" no-go results for the sphere relatively easily (as compared to $\mathbf{R}^{2n}$).

4.3 T^2

At the end of the previous subsection, we showed that there was a full quantization of the sphere, albeit a "trivial" one. Here we exhibit a nontrivial full quantization of the Poisson algebra of the torus. Proofs for the results in this section can be found in [**Go3**].

Consider the torus T^2 thought of as $\mathbf{R}^2/\mathbf{Z}^2$, with symplectic form

$$\omega = B\, dx \wedge dy.$$

We will study the family

$$\mathcal{B}_k = \text{span}\{1, \sin 2\pi kx, \cos 2\pi kx, \sin 2\pi ky, \cos 2\pi ky\}$$

of basic sets with k a positive integer. The crucial difference between this example and the previous ones is that the Poisson algebras $\wp(\mathcal{B}_k)$ are all infinite-dimensional.

Now (T^2, ω) is quantizable provided $B = N\hbar$ for some nonzero integer N. Fix $N = 1$ and let L be the corresponding Kostant-Souriau prequantization line bundle over T^2 [**Ki**]. Then one can identify the space of smooth sections $\Gamma(L)$ with the space of "quasi-periodic" functions $\varphi \in C^\infty(\mathbf{R}^2)$ satisfying

$$\varphi(x+m,\, y+n) = e^{2\pi i m y}\varphi(x,\, y)\,, \quad n,\, m \in \mathbf{Z},$$

and the prequantization Hilbert space $\mathcal{H}$ with the (completion of) the set of those quasi-periodic φ which are L^2 on $[0,\,1) \times [0,\,1)$. The associated prequantization map $\mathcal{Q} : \mathcal{P} \to \text{Op}\big(\Gamma(L)\big)$ (for a specific choice of connection on L) is defined by

$$\mathcal{Q}(f) = -i\hbar\left[\frac{\partial f}{\partial x}\left(\frac{\partial}{\partial y} - \frac{i}{\hbar}x\right) - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right] + f.$$

As the torus is compact, these operators are essentially self-adjoint on $\Gamma(L) \subset \mathcal{H}$.

Theorem 9 *$\mathcal{Q}$ is a quantization of $(\mathcal{P}, \mathcal{B}_k)$ for all positive integers k.*

Proof: Since $\mathcal{Q}$ is a prequantization, it suffices to verify (Q4) and (Q5). To this end it is convenient to use complex notation and write

$$\mathcal{B}_k = \text{span}\big\{1, e^{\pm 2\pi i k x}, e^{\pm 2\pi i k y}\big\}\,.$$

The analysis is simplified by applying the Weil-Brezin-Zak transform Z [**Fo**, §1.10] to the above data. Define a unitary map $Z : \mathcal{H} \to L^2(\mathbf{R})$ by

$$(Z\phi)(x) = \int_0^1 \phi(x,y)\,dy$$

with inverse

$$(Z^{-1}\psi)(x,y) = \sum_{m\in\mathbf{Z}} \psi(x+m)e^{-2\pi i m y}.$$

Under Z the domain $\Gamma(L)$ maps onto the Schwartz space $\mathcal{S}(\mathbf{R})$ [**Ki**]. Setting $A_\pm := Z\mathcal{Q}(e^{\pm 2\pi i k x})Z^{-1}$ and $B_\pm := Z\mathcal{Q}(e^{\pm 2\pi i k y})Z^{-1}$ we compute, as operators on $\mathcal{S}(\mathbf{R})$,

$$\begin{aligned}(A_\pm\psi)(x) &= e^{\pm 2\pi i k x}(1 \mp 2\pi i k x)\psi(x)\\ (B_\pm\psi)(x) &= \left(1 \mp 2\pi\hbar k\frac{d}{dx}\right)\psi(x \pm k).\end{aligned}$$

Then $A_\pm{}^* = \overline{A_\mp}$ on the domain $\{\psi \,|\, x\psi \in L^2(\mathbf{R})\}$, and likewise $B_\pm{}^* = \overline{B_\mp}$ on $\{\psi \,|\, d\psi/dx \in L^2(\mathbf{R})\}$.[9] In fact $\overline{A_\pm}$ and $\overline{B_\pm}$ are normal operators.

To show that $\mathcal{Q}(\mathcal{B}_k)$ is an irreducible set, let us suppose that T is a bounded s.a. operator on $L^2(\mathbf{R})$ which strongly commutes with $\overline{A_\pm}$ and $\overline{B_\pm}$. Then T must commute (in the weak sense) with these operators on their respective domains.[10] Consequently T commutes with both

$$\overline{A_-}\,\overline{A_+} = I + 4\pi^2k^2x^2$$

on the domain $\{\psi \,|\, x^2\psi \in L^2(\mathbf{R})\}$, and

$$\overline{B_-}\,\overline{B_+} = I - 4\pi^2\hbar^2k^2\frac{d^2}{dx^2}$$

on $\{\psi \,|\, d^2\psi/dx^2 \in L^2(\mathbf{R})\}$. From these equations we see that T commutes, and therefore strongly commutes, with the closures of two of the three generators of the metaplectic representation (7) of $\mathrm{sp}(2,\mathbf{R})$ on $\mathcal{S}(\mathbf{R})$.

Let μ denote the metaplectic representation of the metaplectic group Mp(2,$\mathbf{R}$) on $L^2(\mathbf{R})$. We have in effect just established that T commutes with the one parameter groups $\exp\big(is\,\overline{x^2}\big)$ and $\exp\big(-it\hbar^2\,\overline{d^2/dx^2}\big)$. Now classically the exponentials $\exp(sx^2)$ and $\exp(ty^2)$ generate $\mathrm{Sp}(2,\mathbf{R})$ [**GS**, §4]. As $\mathrm{Mp}(2,\mathbf{R}) \to \mathrm{Sp}(2,\mathbf{R})$ is a double covering, the corresponding exponentials in Mp(2,$\mathbf{R}$) generate a neighborhood of the identity in the metaplectic group. Since $\mu\big[\exp(sx^2)\big] = \exp\big(is\,\overline{x^2}\big)$ and $\mu\big[\exp(ty^2)\big] = \exp\big(-it\hbar^2\,\overline{d^2/dx^2}\big)$, it follows that T commutes with $\mu(\mathcal{M})$ for all $\mathcal{M}$ in a neighborhood of the identity in Mp(2,$\mathbf{R}$) and hence, as this group is connected, for all $\mathcal{M} \in \mathrm{Mp}(2,\mathbf{R})$.

Although the metaplectic representation μ is reducible, the subrepresentations μ_e and μ_o on each invariant summand of $L^2(\mathbf{R}) = L^2_e(\mathbf{R}) \oplus L^2_o(\mathbf{R})$ of even and odd functions are irreducible [**Fo**, §4.4]. Writing $T = P_eT + P_oT$, where P_e and P_o are the even and odd projectors, one has

$$[P_eT, \mu(\mathcal{M})] = 0 \tag{17}$$

for any $\mathcal{M} \in$ Mp(2,$\mathbf{R}$). It then follows from the irreducibility of the subrepresentation μ_e that $P_eT = k_eP_e + RP_o$ for some constant k_e and some operator $R : L^2_o(\mathbf{R}) \to L^2_e(\mathbf{R})$. Substituting this expression into (17) yields $[RP_o, \mu(\mathcal{M})] = 0$, and Schur's Lemma then implies that R is either an isomorphism or is zero. But R cannot be an isomorphism as the representations μ_e and μ_o are inequivalent [**Fo**, Thm. 4.56]. (Recall that two unitary representations are similar iff they are unitarily equivalent.) Thus $P_eT = k_eP_e$. Similarly $P_oT = k_oP_o$, whence $T = k_eP_e + k_oP_o$.

But now a short calculation shows that T commutes with

[9] $d\psi/dx$ is to be understood in the sense of tempered distributions.

[10] Here and in what follows we use the fact that a bounded operator weakly commutes with an (unbounded) normal operator iff they strongly commute.

$$\overline{A_+} - \overline{A_-} = 2i(\sin 2\pi kx - 2\pi kx \cos 2\pi kx)$$

only if $k_e = k_o$. Thus T is a multiple of the identity, and so $\{A_\pm, B_\pm\}$ is an irreducible set, as was to be shown. Thus (Q4) is satisfied.

For (Q5), we claim that the linear span of the Hermite functions form a dense set of separately analytic vectors for the e.s.a. components of $\{A_\pm, B_\pm\}$. From the expression above for $A_\pm$, it is clear that a vector will be analytic for the e.s.a. components of $A_\pm$ iff it is analytic for multiplication by x. But it is well known that the Hermite functions are analytic for this latter operator. The result for $B_\pm$ is obtained directly from this by means of the Fourier transform. □

Remark: 10. It is interesting to note that the proof breaks down when $|N| \neq 1$ [**Go3**]. While it is not known to what extent this theorem will remain valid in general (but see §5), one can prove in the special case when $k = |N|$ that the corresponding prequantization map does *not* represent $\mathcal{B}_N$ irreducibly, and so does not qualify as a quantization of $\big(\mathcal{P}(T^2, \omega_N), \mathcal{B}_N\big)$.

5 Speculations

Comparing the behavior of the examples presented in the previous section, we see that we were able to obtain a very sharp no-go theorem for S^2, a relatively weaker no-go result for $\mathbf{R}^{2n}$, and no obstruction at all in the case of T^2. On the basis of these examples, we attempt to extract the key features which govern the appearance of obstructions to a full quantization as well as their severity.

Of course, any conclusions that we can draw at this point are necessarily quite tentative, due to the paucity of examples against which to test them. There are also various aspects of our three main examples that are still not well understood. Nonetheless, some interesting observations can be made, which may prove helpful in subsequent investigations.

Of our three examples, the torus is clearly much different than the others. It is not a Hamiltonian homogeneous space, and the Poisson algebras generated by the basic sets $\mathcal{B}_k$ are infinite-dimensional. The latter property seems to be the controlling factor: In effect, since $\wp(\mathcal{B}_k)$ is infinite-dimensional, the irreducibility requirement (Q4) loses much of its force – so much so, in the case of the torus, that it precludes the existence of an obstruction. Hence we propose that a general Groenewald-Van Hove theorem takes the form:

Conjecture 1 *Let M be a symplectic manifold and $\mathcal{B} \subset \mathcal{P}(M)$ a basic set with $\wp(\mathcal{B})$ finite-dimensional. Then there is no nontrivial strong quantization of $\big(\mathcal{P}(M), \mathcal{B}\big)$.*

A distinction must be made here depending upon whether $\wp(\mathcal{B})$ is a "compact" Lie algebra (i.e., is the Lie algebra of a compact Lie group)[11] or not. The

[11] This is nonstandard terminology.

reason is that by (Q4) and (Q5) the representation $\mathcal{Q}\big(\wp(\mathcal{B})\big)$ will be integrable to an irreducible unitary representation $\varPi$ of the connected, simply connected Lie group $\tilde{G}$ whose Lie algebra is $\wp(\mathcal{B})$. If $\wp(\mathcal{B})$ is compact, then some discrete quotient G of $\tilde{G}$ will be compact. Let Γ be the kernel of the covering map $\tilde{G} \to G$; then Γ is central in $\tilde{G}$. By irreducibility, $\varPi(\gamma)$ must be a multiple of the identity for each $\gamma \in \Gamma$ and so, by unitarity, each $\varPi(\gamma)$ has unit modulus. It follows that $\varPi$ can be viewed as a projective representation of G. This projective representation, in turn, can be realized as an honest representation of a certain central extension of G by $\varPi(\Gamma)$ [**BaRa**, §13.2]. Since $\varPi(\Gamma) \subset T^1$ and is discrete, it is finite, and hence this central extension is compact. Consequently the irreducible representations of the latter are finite-dimensional, and so all functional analytic difficulties are obviated. In the compact case, then, one can delete the adjective "strong" in Conjecture 1. This explains (in part) why it was "easier" to obtain the no-go result for the sphere than for $\mathbf{R}^{2n}$.

In particular, generalizing the analysis of S^2 given in [**GGH**], it appears likely that $\mathbf{C}P^n$ with basic set $\mathrm{u}(n)$ will admit no nontrivial full quantizations.

When $\wp(\mathcal{B})$ is noncompact it is necessary to confront these analytic difficulties. In fact, it is the noncompactness of $\mathrm{h}(2n)$ that is responsible for the possible non-integrability of $\mathcal{Q}(P^2)$, and hence the splitting of the no-go results for $\mathbf{R}^{2n}$ into "weak" and "strong" cases. In this regard, it would be very interesting to determine whether there exists a full *weak* quantization of $\big(\mathcal{P}(\mathbf{R}^{2n}), \mathrm{h}(2n)\big)$, or at least a weak quantization of (P, P^1). Of course, $\mathcal{Q}(P^2)$ cannot then be integrable, and the Von Neumann rules (8) cannot hold. The physical import of this is discussed in [**KLZ**].

As all extant no-go results are for elementary systems, it would be useful to study symplectic manifolds which are not Hamiltonian homogeneous spaces, but which nonetheless have basic sets with $\wp(\mathcal{B})$ finite-dimensional. A particularly interesting example of this type is T^*S^1, with $\mathcal{B} = \mathrm{span}\{1, \sin\theta, \cos\theta, l\}$, where (θ, l) are canonical coordinates with l being the angular momentum. It lies "halfway" between $\mathbf{R}^2$ and T^2, and so should provide a good test of Conjecture 1. Other examples worth studying are various elementary systems for $\mathrm{Sp}(2n, \mathbf{R}) \times \mathbf{R}$, such as $\mathbf{R}^{2n} \setminus \{\mathbf{0}\}$ and its coadjoint orbits.[12]

In view of the torus example it seems equally reasonable to propose

Conjecture 2 *Let M be a symplectic manifold and $\mathcal{B}$ a basic set with $\wp(\mathcal{B})$ dense in $C^\infty(M)$. Then there exists a nontrivial quantization of $\big(\mathcal{P}(M), \mathcal{B}\big)$.*

Obviously, a necessary condition for $\mathcal{Q}$ to be a full quantization of $(\mathcal{P}, \mathcal{B})$ is that $\mathcal{Q}$ represent $\mathcal{P}$ itself irreducibly. It turns out [**Ch3**, **Tu**] that this is so for all Kostant-Souriau prequantizations[13]; thus it is natural to consider the case

[12] The factor $\mathbf{R}$ is inserted here so that the corresponding basic set will include the constants, cf. (B4).

[13] However, it should be noted that there are other prequantizations which do not represent $\mathcal{P}$ irreducibly, for instance, the prequantization of Avez [**Av**, **Ch2**].

when M is prequantizable in this sense. In fact, in this context [**Tu**] gives even more:

Proposition 10 *Let M be an integral symplectic manifold, L a Kostant-Souriau prequantization line bundle over M and $\mathcal{Q}_L$ the corresponding prequantization map. Let $\mathcal{B}$ be a basic set with $\wp(\mathcal{B})$ dense in $C^\infty(M)$. Then $\mathcal{Q}_L$ represents $\mathcal{B}$ irreducibly on the domain which consists of compactly supported sections of L.*

Set $D_0 = \Gamma(L)_0$, the compactly supported sections of L. By construction $\mathcal{Q}_L : \mathcal{P}(M) \to \mathrm{Op}(D_0)$ satisfies (Q1)-(Q3). This proposition states that $\mathcal{Q}_L$ satisfies (Q4) as well. Thus to obtain a full quantization it remains to verify (Q5) – perhaps on some appropriately chosen coextensive domain D; unfortunately, it does not seem possible to do this except in specific instances. A first test would be to understand what happens for $\big(\mathcal{P}(T^2, \omega_N), \mathcal{B}_1\big)$. In any event, Proposition 10 does provide a certain amount of support for Conjecture 2.

The "gray area" between these two conjectures consists of symplectic manifolds with basic sets $\mathcal{B}$ for which $\wp(\mathcal{B})$ is infinite-dimensional, yet not dense in $C^\infty(M)$. That the infinite-dimensionality of $\wp(\mathcal{B})$ alone may be enough to guarantee the existence of a full quantization is evinced by our results for $\big(\mathcal{P}(T^2, \omega_1), \mathcal{B}_k\big)$ with $k > 1$. (See however Remark **10**.)

Proving Conjecture 1 in any sort of generality seems well beyond what is possible with present technology. Still, based on the examples that have already been worked out, we can suggest an avenue of attack: Assume that there does exist a full quantization of $(\mathcal{P}(M), \mathcal{B})$. The irreducibility and integrability requirements upon $\mathcal{Q}(\mathcal{B})$ should lead to Von Neumann rules for elements of $\mathcal{B}$. For example, in the case of $\mathbf{R}^{2n}$ these rules are:

$$\mathcal{Q}\big(x^2\big) = \mathcal{Q}(x)^2, \quad x \in \mathrm{h}(2n).$$

Similarly, for the sphere, we computed the somewhat more complicated rules

$$\mathcal{Q}\big({S_i}^2\big) = a\mathcal{Q}(S_i)^2 + cI, \quad i = 1,\, 2,\, 3$$

with $ac \neq 0$. The idea is then to show that eventually, the Von Neumann rules lead to a contradiction with the Poisson bracket $\to$ commutator rule. The main difficulty with this approach is that the derivation of these rules requires a detailed knowledge of the representation theory of $\wp(\mathcal{B})$. Perhaps it is possible to streamline this process, at least for certain types of basic sets. Another problem is that one doesn't know where to look for the contradiction. For $\mathbf{R}^{2n}$, it arises when one considers cubic polynomials, while for S^2 the contradiction comes sooner, when one considers spherical harmonics of degree 2. Fortunately, another conjecture (see below) enables one to "guess" where the contradiction might lie. It will likely prove necessary to work through a few more examples of Groenewold-Van Hove obstructions before one is able to refine this approach enough to become workable.

As well, it would be useful to consider basic sets other than the ones we have studied in the standard examples, for instance, the Euclidean algebras that arise in optics [**GS**, §17]. We have also restricted consideration to polynomial subalgebras to a large extent, but there are other subalgebras $\mathcal{O}$ which are of interest (e.g., on $\mathbf{R}^{2n}$, those functions which are constant outside some compact set [**Ch2**]).

A negative answer to the conjecture might indicate that one should strengthen the conditions defining a basic set by, e.g., replacing (B3) by (C2) as discussed in §3. Or, if the conjecture seems undecidable, perhaps one should abandon the definition of a quantization map solely in terms of basic sets and consider an alternative. However, the two other ways to define a quantization map listed previously suffer from serious flaws. If one imposes Von Neumann rules at the outset, then one tends to run into difficulties rather quickly – especially if one tries to enforce the rules on all of $\mathcal{P}(M)$ and not some basic subset thereof – as was shown in §4.1. Furthermore, it is unclear what form Von Neumann rules should take in general, as is illustrated by the unintuitive rules for the sphere exhibited above. For instance, mimicking the situation for $\mathbf{R}^{2n}$, one might simply postulate that $\mathcal{Q}(x^2) = \mathcal{Q}(x)^2$ for $x \in u(2)$.[14] But the squaring rule for angular momentum is plainly incompatible with the (presumably) correct Von Neumann rules (14), because of the restriction that $ac \neq 0$.[15] Even disregarding this point, one would still "miss" various possibilities (corresponding to the freedom in the choice of parameters a, c), which do occur in specific representations. And in the case of the torus, Von Neumann rules are effectively moot, since the explicit prequantization map $\mathcal{Q}$ itself determines the quantization of every observable. Another way of phrasing this is that the irreducibility requirement is so weak in this context that it cannot enforce any Von Neumann rules, and hence does not lead to a contradiction with the given $\mathcal{Q}$. All in all, it appears as if the Von Neumann rules play a secondary role; the basic set $\mathcal{B}$ is the primary object. It is also more compelling physically and pleasing æsthetically to require $\mathcal{Q}$ to satisfy an irreducibility requirement than a Von Neumann rule.

There are problems with the polarization approach as well. For one thing, symplectic manifolds need not be polarizable [**Go2**]. This relatively rare occurrence notwithstanding, there are quantizations which cannot be obtained by polarizing a prequantization: a well known example is the extended metaplectic quantization of the pair $\big(\mathrm{hsp}(2n,\mathbf{R}),\mathrm{h}(2n)\big)$ [**Bl2**]. As we shall see presently, the specific predictions of geometric quantization theory are also off the mark in a number of instances.

Finally, it should be mentioned that these three approaches to quantization typically lead to obstructions in one way or another. We have already seen in the beginning of §4.1 that the imposition of Von Neumann rules, in and by themselves, results in inconsistencies; moreover, Von Neumann rules played a crucial

[14] In effect, this is exactly what transpired in the case of $\mathbf{R}^{2n}$: condition (Q5$'$) was specifically designed so as to produce the squaring rule for elements of $\mathrm{h}(2n)$.

[15] Because of this, [**KLZ**] would refer to (14) as "*non*-Neumann rules"!

role in deriving the Groenewold-Van Hove obstructions for $\mathbf{R}^{2n}$ and S^2. In the context of polarizations, it is known that the only observables which are consistently quantizable *a priori* are those whose Hamiltonian vector fields preserve a given polarization [**Bl1**, **Wo**]. While this does not preclude the possibility of quantizing more general observables, attempts to quantize observables outside this class in specific examples usually result in inconsistencies. In *all* instances, the set of *a priori* quantizable observables relative to a given polarization forms a proper subalgebra of the Poisson algebra of the given symplectic manifold. This observation provides further corroboration that Groenewold-Van Hove obstructions to quantization should be ubiquitous.

Setting aside the question of the existence of Groenewold-Van Hove obstructions, let us now suppose that there is such an obstruction, so that it is impossible to consistently quantize all of $\mathcal{P}$. The question is: what are the maximal subalgebras $\mathcal{O} \subset \mathcal{P}$ containing the given basic set $\mathcal{B}$ such that $(\mathcal{O}, \mathcal{B})$ can be quantized? Now, technical issues aside, given a representation $\mathcal{Q}$ of $\wp(\mathcal{B})$ on a Hilbert space $\mathcal{H}$, one ought to be able to induce a representation of its Poisson normalizer $\mathcal{N}\big(\wp(\mathcal{B})\big)$ on $\mathcal{H}$. (Indeed, the structure $\big(\mathcal{N}(\wp(\mathcal{B})), \mathcal{B}\big)$ brings to mind an infinitesimal version of a Mackey system of imprimitivity [**BaRa**].) In particular, for $\mathbf{R}^{2n}$ one has that $\mathcal{N}\big(\mathrm{h}(2n)\big) = \mathrm{hsp}(2n, \mathbf{R})$, and for S^2 one computes that $\mathcal{N}\big(\mathrm{u}(2)\big) = \mathrm{u}(2)$. In both cases, we have shown that these normalizers are in fact the maximal subalgebras that can be consistently quantized. Thus it seems reasonable to assert:

Conjecture 3 *Let $\mathcal{B}$ be a basic set, with $\wp(\mathcal{B})$ finite-dimensional. Then every integrable irreducible representation of $\wp(\mathcal{B})$ can be extended to a quantization of $\big(\mathcal{N}(\wp(\mathcal{B})), \mathcal{B}\big)$. Furthermore, no nontrivial quantization of $\big(\mathcal{N}(\wp(\mathcal{B})), \mathcal{B}\big)$ can be extended beyond $\mathcal{N}\big(\wp(\mathcal{B})\big)$.*

If true, this conjecture would point where to look for a Groenewold-Van Hove contradiction, viz. just outside the normalizer. This is in exact agreement with the examples. However, it should be emphasized that there may exist subalgebras $\mathcal{O} \subset \mathcal{P}$, with $(\mathcal{O}, \mathcal{B})$ maximally quantizable, which do not arise in this fashion. A familiar example is the position subalgebra $S = \big\{f(q)p + g(q)\big\}$ of $\mathcal{P}(\mathbf{R}^2)$ encountered in §4.1. It is not clear how one could "discover" this subalgebra given just the basic set $\mathcal{B} = \mathrm{h}(2)$ (but see below); note also that $\mathcal{N}\big(\mathrm{h}(2)\big) \not\subset S$.

This is reminiscent of the situation in geometric quantization with respect to polarizations. Suppose that $\mathcal{A}$ is a polarization of $\mathcal{P}_{\mathbf{C}}(M)$. Then one knows that one can consistently quantize those observables which preserve $\mathcal{A}$, i.e., which belong to the real part of $\mathcal{N}(\mathcal{A})$ [**Bl1**, **Wo**]. In this way one obtains a "lower bound" on the set of quantizable functions for a given polarization. If one takes the antiholomorphic polarization on S^2, then it turns out that the set of *a priori* quantizable functions obtained in this manner is precisely the u(2) subalgebra span$\{1, S_1, S_2, S_3\}$. But it may happen that the real part of $\mathcal{N}(\mathcal{A})$ is too small,

as happens for $\mathbf{R}^{2n}$ with the antiholomorphic polarization. In this case the real part of $\mathcal{N}(\mathcal{A})$ is only a proper subalgebra of P^2, and in particular is not maximal. This illustrates the fact, alluded to previously, that the extended metaplectic representation cannot be derived via geometric quantization. Furthermore, in the case of the torus, introducing a polarization will drastically cut down the set of *a priori* quantizable functions, which is at odds with the existence of a full quantization of this space. So geometric quantization is not a reliable guide insofar as computing maximally quantizable subalgebras of observables. On the other hand, the position subalgebra $\mathcal{S} = \{f(q)p + g(q)\}$ is just the normalizer of the vertical polarization $\mathcal{A} = \{h(q)\}$ on $\mathbf{R}^{2n}$, so this subalgebra finds a natural interpretation in the context of polarizations.

Clearly, there must be some connection between polarizations and basic sets that awaits elucidation. It would be interesting to determine if there is a way to recast the Groenewold-Van Hove results in terms of polarizations. It would also be worthwhile, assuming Conjecture 3 to be correct, to see whether one can use the *ab initio* knowledge of a maximal set of quantizable observables to refine geometric quantization theory, or to develop a new quantization procedure, which is adapted to the Groenewold-Van Hove obstruction in that it will automatically be able to quantize this maximal set.

One of our goals in this paper was to obtain results which are independent of the particular quantization scheme employed, as long as it is Hilbert-space based. Therefore it is interesting that the go and no-go results described in this proposal have direct analogues in deformation quantization theory, since this theory was developed, at least in part, to avoid the use of Hilbert spaces altogether [**BFFLS**]. So for example, the no-go result for S^2 is mirrored by the fact that there are no strict SU(2)-invariant deformation quantizations of $\mathcal{P}(S^2)$ [**Ri1**], while the go theorem for T^2 has as a counterpart the result that there do exist strict deformation quantizations of the torus [**Ri1**]. It is generally believed that the existence of Groenewold-Van Hove obstructions necessitates a weakening of the Poisson bracket $\rightarrow$ commutator rule (by insisting that it hold only to order $\hbar$), but the analogies above indicate that this may not suffice to remove the obstructions. There are undoubtedly many important things to be learned by getting to the heart of this phenomenon.

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